MATH 312
Lecture 15

ORTHOGONAL BASES & MATRICES

Recently we've been discussing PROJECTIONS of vectors $\vec{w}$ onto subspaces V, starting with a BASIS $\vec{v}_1, \ldots, \vec{v}_k$ of V.

- Set $A = [\vec{v}_1 \cdots \vec{v}_k]$, an $n \times k$ matrix
- $$P_V \vec{w} = \Big[A \underbrace{(A^T A)^{-1}}_{\text{Hard to compute}} A^T\Big] \vec{w}$$

Also, the method of LEAST SQUARES to find "best" answers when no solution exists for $A\vec{x} = \vec{b}$. We try to solve the "normal equations"

$$A^T A \vec{x} = A^T \vec{b}$$

$$\text{or,} \quad \vec{x} = \underbrace{(A^T A)^{-1}}_{\text{still hard to compute}} A^T \vec{b}$$

Okay, so when is $(A^T A)^{-1}$ EASY to compute? For the inverse to exist, we assume that A is $n \times k$ and has rank k, with $k \le n$. Again let's write

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{bmatrix}$$

So

$$A^T = \begin{bmatrix} \text{---} & \vec{v}_1^T & \text{---} \\ \text{---} & \vdots & \text{---} \\ \text{---} & \vec{v}_k^T & \text{---} \end{bmatrix}$$

So,

$$A^T A = \begin{bmatrix} — v_1^T — \\ \vdots \\ — v_k^T — \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_k \\ v_2^T v_1 & v_2^T v_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ v_k^T v_1 & \cdots & \cdots & v_k^T v_k \end{bmatrix}$$

The <u>NICEST</u> matrix this could possibly be (while remaining invertible!) is the identity!

So, we'd like the columns $\vec{v}_1, \ldots, \vec{v}_k$ to satisfy:

$$\boxed{v_i^T v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}}$$

That is, we'd like for the $\vec{v}_i$'s to be

- UNIT VECTORS, and
- PAIRWISE ORTHOGONAL.

Such collections $\{v_1, \ldots, v_k\}$ of vectors are called <u>ORTHONORMAL</u>. Matrices with orthonormal columns are called <u>ORTHOGONAL</u> <u>MATRICES</u>.

$$\boxed{Q \text{ is an orthogonal matrix} \iff Q^T Q = \text{identity}}$$

Examples

1. Rotation matrices: $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

2. Permutation matrices: $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

AMAZING BENEFITS OF ORTHOGONAL MATRICES (OR ORTHONORMAL BASES).

1. If we want to project a vector $\vec{w}$ onto a subspace V with an orthonormal basis $\{\vec{q}_1, \dots, \vec{q}_k\}$, then set

$$Q = [\vec{q}_1 \cdots \vec{q}_k]$$

and note: $P_V \vec{w} = [Q(\cancel{Q^TQ})^{-1}Q^T]\vec{w}$

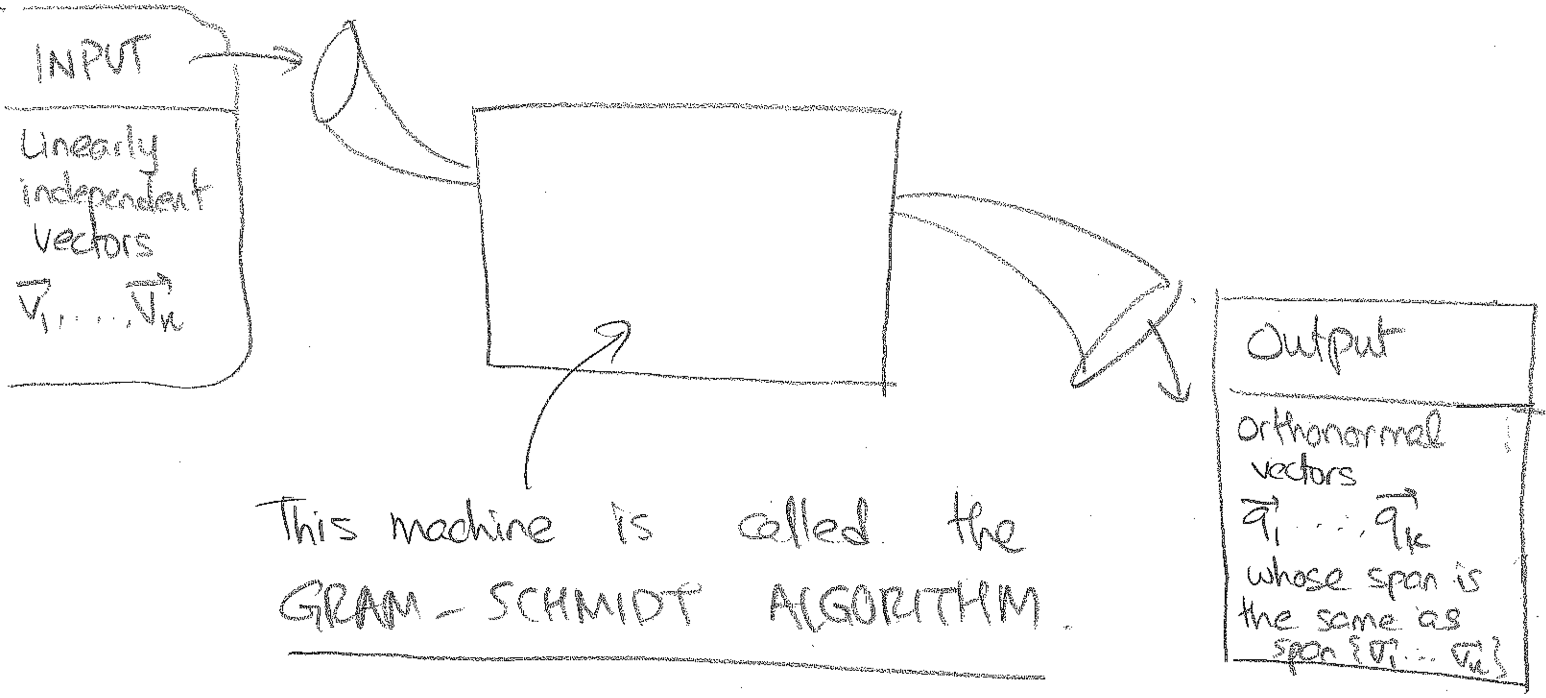
$$= \underline{QQ^T\vec{w}} \leftarrow \text{Nice!}$$

(Quick warning: don't assume QQ^T is also the identity... That only works if Q is square!)

2. If we want to compute the least squares solution to $Q\vec{x} = \vec{b}$ where Q is an orthogonal matrix, we have

$$\vec{x} = (\cancel{Q^TQ})^{-1}Q^T\vec{b} = \underline{Q^T\vec{b}} \leftarrow \text{Ha!}$$

We want a machine that does this:

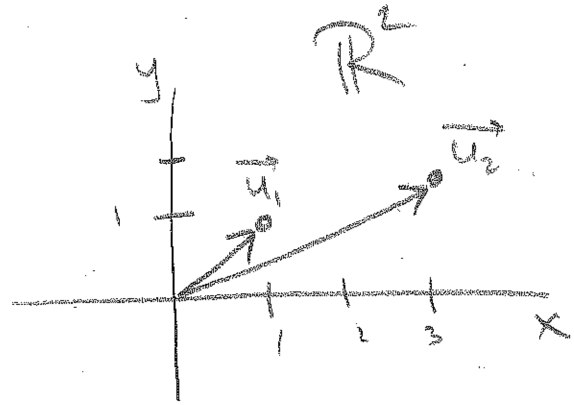


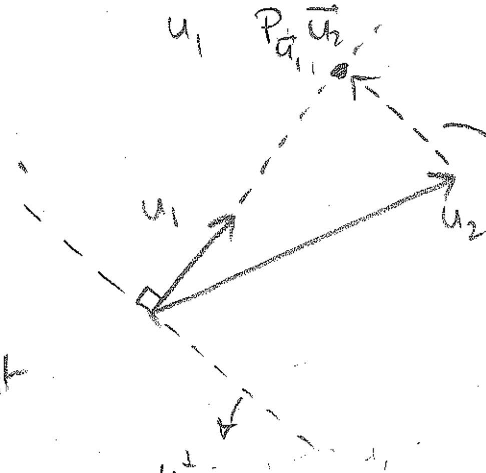
This machine is called the <u>GRAM-SCHMIDT ALGORITHM</u>.

(<u>START</u> <u>SIMPLE</u> : $k=2$).

eg: $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$.

u_1 $\quad$ u_2





Project $\vec{u}_2$ onto $\vec{u}_1$.
This gives the component of u_2 $P_{\vec{u}_1}\vec{u}_2$ PARALLEL to $\vec{u}_1$.

So, the difference $(u_2 - P_{u_1} u_2)$ is perpendicular to $\vec{u}_1$.

Note:
$P_{\vec{u}_1}\vec{u}_2$ is just
$\left[\frac{u_1^T u_2}{u_1^T u_1}\right] u_1$

So, $\vec{u}_1$ and $u_2 - P_{\vec{u}_1}\vec{u}_2$ are orthogonal: divide by their lengths to get orthonormal vectors.

RINSE AND REPEAT! (Gram-Schmidt Algorithm)

Given $\{u_1, \dots, u_k\}$ linearly independent, (vectors)

Set $v_1 = u_1$

Set $v_2 = u_2 - \left[\dfrac{v_1^T u_2}{v_1^T v_1}\right] v_1$

Set $v_3 = u_3 - \left[\dfrac{v_1^T u_3}{v_1^T v_1}\right] v_1 - \left[\dfrac{v_2^T u_3}{v_2^T v_2}\right] v_2$

$\vdots$

Set $v_k = u_k - \left[\dfrac{v_1^T u_k}{v_1^T v_k}\right] v_1 - \cdots - \left[\dfrac{v_{k-1}^T u_k}{v_{k-1}^T v_{k-1}}\right] v_{k-1}$

Finally, set $q_1 = v_1 / \|v_1\|$, $q_2 = v_2 / \|v_2\|$ ... etc.

Eg Produce an orthonormal basis $\{q_1, q_2, q_3\}$ with the same span as

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

GS:

- $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

- $v_2 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} - \dfrac{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$\boxed{\text{Check: } v_2^T v_1 = 0}$ $\longrightarrow$

$$\circ \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{[1\ 1\ 0]\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}{[1\ 1\ 0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{[1\ 1\ -3]\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}{[1\ 1\ -3]\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{-5}{11}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} -5/11 \\ -5/11 \\ +15/11 \end{bmatrix}$$

$\boxed{\text{check: } v_3^T v_1 = 0,\ v_3^T v_2 = 0}$

$=$ ugh...

Finally: $q_1 = v_1 / \|v_1\| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$q_2 = v_2 / \|v_2\| = \frac{1}{\sqrt{11}}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

$q_3 =$ more ugh...

QR DECOMPOSITION

Note a triangular system: (in GS algorithm)

$u_1 = v_1,$

$u_2 = v_2 + (\circ) v_1$

$u_3 = v_3 + (\circ) v_1 + (\circ) v_2.$ and so on

So,

$$\underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} | & & | \\ q_1 & \cdots & q_k \\ | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} q_1^T u_1 & q_1^T u_2 & \cdots & q_1^T u_k \\ 0 & q_2^T u_2 & & \\ \vdots & 0 & \ddots & \\ 0 & & 0 & q_k^T u_k \end{bmatrix}}_{R}$$